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Semi-Cohen Boolean algebras¹

Bohuslav Balcar^{a,2}, Thomas Jech^{b,*,2}, Jindřich Zapletal^c^a *Mathematical Institute of the Academy of Sciences of Czech Republic, Žitná 25, Praha 1, Czech Republic*^b *Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA*^c *Division of Mathematics and Astronomy, California Institute of Technology, Pasadena, CA 91125, USA*

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Abstract

We investigate classes of Boolean algebras related to the notion of forcing that adds Cohen reals. A *Cohen algebra* is a Boolean algebra that is dense in the completion of a free Boolean algebra. We introduce and study generalizations of Cohen algebras: semi-Cohen algebras, pseudo-Cohen algebras and potentially Cohen algebras. These classes of Boolean algebras are closed under completion.

1. Introduction

For an infinite cardinal κ , let C_κ denote the complete Boolean algebra that adjoins κ Cohen reals. C_κ is the completion of the free Boolean algebra on κ generators; equivalently, C_κ is the algebra of all regular open subsets of the topological product space $\{0,1\}^\kappa$. We call a Boolean algebra B a *Cohen algebra* if the completion of B is C_κ .

We investigate Boolean algebras that closely resemble Cohen algebras, particularly the class of algebras called *semi-Cohen*. These algebras were introduced in 1992 by Fuchino and Jech, motivated by Koppelberg's work on Cohen algebras [10]. The work on this project was done between 1992 and 1995 during Balcar's visits at Penn State and Jech's visits in Prague.

* Corresponding author. Tel.: 814 865 0361; e-mail: jech@math.psu.edu.

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² Current address: Center for Theoretical Study, Jilská 1, 110 00 Praha 1, Czech Republic.

Semi-Cohen (called regularly filtered) and related Boolean algebras are also the subject of a recent monograph by Heindorf and Shapiro [7]. Their work deals with generalizations of projectivity and uses algebraic, rather than set theoretic, methods and point of view.

A game property equivalent to semi-Cohen also appears in [4].

An application of semi-Cohen algebras in topological dynamics appears in [1].

Definition 1.1. A Boolean algebra B of uniform density κ is *semi-Cohen* if $[B]^\omega$ has a closed unbounded set of countable regular subalgebras of B .

Every Cohen algebra is semi-Cohen. In Section 3 we prove the following characterization of Cohen algebras, a slight improvement of results due to Koppelberg [10] and Bandlow [2]:

Theorem 3.2. *Let κ be an uncountable cardinal and let B be a Boolean algebra of uniform density κ . Then B is a Cohen algebra if and only if the set*

$$\{A \in [B]^\omega : A \leq_{\text{reg}} B\}$$

contains a closed unbounded set C with the property

$$\text{if } A_1, A_2 \in C \text{ then } \langle A_1 \cup A_2 \rangle \in C,$$

where $\langle A_1 \cup A_2 \rangle$ is the subalgebra of B generated by $A_1 \cup A_2$.

It is not difficult to see that every regular subalgebra, of uniform density, of a semi-Cohen algebra is itself semi-Cohen (cf. Theorem 4.1). In particular, every complete subalgebra of uniform density of \mathbf{C}_κ is semi-Cohen. Thus, the concept of semi-Cohen algebras is relevant to the following problems (cf. [9, 8]):

Is every complete subalgebra of \mathbf{C}_κ of uniform density isomorphic to some \mathbf{C}_λ ?

This problem was recently solved by Koppelberg and Shelah in [11] and we return to it below.

In Section 4 we investigate semi-Cohen algebras. Among the results of Section 4 are these (some have also been proved by Fuchino):

Theorem 4.3. *Let B be a Boolean algebra of uniform density. The following are equivalent:*

- (a) *B is a semi-Cohen algebra.*
- (b) *$V^P \models B$ is a Cohen algebra, where P is the σ -closed collapse of $|B|$ onto \aleph_1 .*
- (c) *There exists a proper forcing P such that $V^P \models B$ is Cohen.*
- (d) *The second player has a winning strategy in the infinite game \mathcal{G} in which two players select in turn elements $a_0, b_0, a_1, b_1, \dots$ of B and the second player wins if $\langle \{a_0, b_0, a_1, b_1, \dots\} \rangle \leq_{\text{reg}} B$.*
- (e) [4] *I has a winning strategy in the following game \mathcal{H} : I plays elements a_i of B and II plays elements $b_i \leq a_i$. I wins iff $\sum b_i = 1$.*

Theorem 4.5. *Let $\{B_\alpha: \alpha < \vartheta\}$ be a continuous increasing chain of semi-Cohen Boolean algebras such that $B_\alpha \leq_{\text{reg}} B_{\alpha+1}$ for every α . Then $\bigcup_{\alpha < \vartheta} B_\alpha$ is semi-Cohen.*

Theorem 4.6. (a) *If A and B are semi-Cohen algebras then $A \times B$ is semi-Cohen.*

(b) *If A is semi-Cohen and $V^A \models \dot{B}$ is semi-Cohen then $A * \dot{B}$ is semi-Cohen.*

(c) *If B is semi-Cohen and if W is an extension of the universe then $W \models B$ is semi-Cohen.*

In Section 5 we present some examples:

Theorem 5.2. *There exists a semi-Cohen algebra of density \aleph_2 that cannot be embedded as a regular subalgebra into a Cohen algebra.*

Theorem 5.11. *There exists an increasing ω -chain of Cohen algebras B_n , with $B_n \leq_{\text{reg}} B_{n+1}$ whose union is not a Cohen algebra.*

We also give a simplified proof of the result of Koppelberg and Shelah mentioned above:

Theorem 5.1 (Koppelberg and Shelah [11]). *For every $\kappa \geq \aleph_2$, C_κ has a complete subalgebra of uniform density κ that is not Cohen.*

We mention a related result from [7, 6.3.2]: a construction of an rc-filtered Boolean algebra (hence semi-Cohen) that is not a Cohen algebra.

In Section 6 we consider pseudo-Cohen algebras:

Definition 6.1. A Boolean algebra B of uniform density κ is *pseudo-Cohen* if it has a stationary set of countable regular subalgebras.

Clearly, every semi-Cohen algebra is pseudo-Cohen, and every regular subalgebra of a pseudo-Cohen algebra is pseudo-Cohen (cf. Theorem 6.2). We characterize pseudo-Cohen algebras, prove preservation properties, and give some examples:

Theorem 6.3. *Let B be an algebra of uniform density κ . The following are equivalent:*

(a) *B is a pseudo-Cohen.*

(b) *There exists an \aleph_0 -distributive forcing P such that $V^P \models B$ is Cohen.*

(c) *The first player does not have a winning strategy in the game \mathcal{G} .*

Proposition 6.4. *If B is pseudo-Cohen and W is a proper-forcing extension of the universe then $W \models B$ is pseudo-Cohen.*

Theorem 6.5. *There exists a pseudo-Cohen algebra that is not semi-Cohen.*

In Section 7 we consider a further generalization:

Definition 7.1. A Boolean algebra B of uniform density is *potentially Cohen* if there exists a forcing P preserving \aleph_1 such that $V^P \models B$ is Cohen.

By Theorem 6.2(b), every pseudo-Cohen algebra is potentially Cohen. As for the converse, we present two results:

Theorem 7.2. *The continuum hypothesis implies that every potentially Cohen algebra is pseudo-Cohen.*

Theorem 7.3. *It is consistent (and follows from $MA + \neg CH$) that the measure algebra (which is not pseudo-Cohen) is potentially Cohen.*

Theorems 5.2 and 6.5 were known to Fuchino previously.

2. Preliminaries

For a Boolean algebra B , we denote B^+ the set of all nonzero elements. We use $+$, \cdot and $-$ to denote Boolean-algebraic operations and \leq for the Boolean algebraic ordering (inclusion). Infinite sums and products, when they exist, are denoted Σ and Π . If B is a Boolean algebra and $X \subseteq B$, we denote

$$\langle X \rangle = \bigcap \{A : A \text{ is a subalgebra of } B \text{ and } X \subseteq A\}$$

the subalgebra generated by X . Every element of $\langle X \rangle$ can be written as the sum $p_1 + \dots + p_n$ where each p_i is $\pm x_i \cdot \pm x_2 \cdot \dots \cdot \pm x_k$ with $x_1, \dots, x_k \in X$. If A is a subalgebra of B and $b_1, \dots, b_n \in B$, we let

$$A(b_1, \dots, b_n) = \langle A \cup \{b_1, \dots, b_n\} \rangle.$$

Note that

$$A(b) = \{a_1 \cdot b + (a_2 - b) : a_1, a_2 \in A\}.$$

Definition 2.1. A subalgebra A of B is a *regular subalgebra*

$$A \leq_{\text{reg}} B,$$

if for any $X \subseteq A$, if $\Sigma^A X$ exists then $\Sigma^A X = \Sigma^B X$.

The following equivalences are well known; cf. [9]:

Lemma 2.2. *The following are equivalent:*

- (a) $A \leq_{\text{reg}} B$,
- (b) every maximal antichain in A is maximal in B ,
- (c) for every $b \in B^+$ there exists some $a \in A^+$ such that no $x \in A^+$ exists with the property that $x \leq a$ and $x \cdot b = 0$.

A set $D \subseteq B$ is *dense* in B if for every $b \in B^+$ there exists a $d \in D$ with $0 < d \leq b$. The *density* of B is the least size of a dense subset of B . B has *uniform density* if for every $a \in B^+$, $B \upharpoonright a$ has the same density. (Every Boolean algebra can be decomposed into algebras of uniform density.) For every Boolean algebra B there exists a unique complete Boolean algebra \bar{B} , the *completion* of B , such that B is dense in \bar{B} . The next lemma summarizes some known facts about regular subalgebras:

Lemma 2.3.

- (a) If A is a finite subalgebra of B then $A \leq_{\text{reg}} B$.
- (b) If $A \leq_{\text{reg}} B$ and $B \leq_{\text{reg}} C$ then $A \leq_{\text{reg}} C$.
- (c) If A is a subalgebra of B , B is a subalgebra of C and if $A \leq_{\text{reg}} C$ then $A \leq_{\text{reg}} B$.
- (d) If A is a dense subalgebra of B then $A \leq_{\text{reg}} B$.
- (e) $A \leq_{\text{reg}} B$ if and only if $\bar{A} \leq_{\text{reg}} \bar{B}$.
- (f) If A and B are complete then $A \leq_{\text{reg}} B$ if and only if A is a complete subalgebra of B .
- (g) If $\{A_i\}_{i \in I}$ is a directed system of algebras such that $A_i \leq_{\text{reg}} A_j$ whenever $i \leq j$, and if $B = \bigcup_{i \in I} A_i$, then $A_i \leq_{\text{reg}} B$ for all $i \in I$.

If A is a subalgebra of B and $b \in B$, then the (*upper*) projection of b to A is the smallest element $a \in A$, if it exists, such that $b \leq a$. The projection of b is denoted $pr^A(b)$.

If $pr^A(b)$ exists for all $b \in A$, then $A \leq_{\text{reg}} B$, and the lower projection $pr_A(b)$ exists for all $b \in B$, where $pr_A(b)$ = the largest $a \leq b$ in A .

A set $X \subseteq B$ is *independent* if

$$\pm x_1 \cdot \pm x_2 \cdot \dots \cdot \pm x_n \neq 0$$

for all (distinct) $x_1, \dots, x_n \in X$. A Boolean algebra B is *free* over X if X is independent and $B = \langle X \rangle$. The *free algebra* over X is unique up to isomorphism and will be denoted Fr_X . We note that Fr_X is isomorphic to the set algebra of all clopen sets of the Cantor space $\{0, 1\}^X$. We also note that if $X \subseteq Y$ then $Fr_X \leq_{\text{reg}} Fr_Y$.

If A is an (uncountable) set, we denote by $[A]^\omega$ the set of all countable subsets of A , and by $[A]^{<\omega}$ the set of all finite subsets of A . A set $C \subseteq [A]^\omega$ is *closed unbounded* if C is closed under unions of countable chains and for every $x \in [A]^\omega$ there exists a $y \in C$ with $x \subseteq y$. Some facts about closed unbounded sets:

Lemma 2.4.

- (a) If $C \subseteq [A]^\omega$ is closed unbounded then there exists a function $F : [A]^{<\omega} \rightarrow A$ such that

$$C \supseteq \{x \in [A]^\omega : F(e) \in x \text{ whenever } e \in [x]^{<\omega}\}.$$

- (b) If C is closed unbounded and $D \subseteq C$ is countable and directed under inclusion then $\bigcup D \in C$.

- (c) If $C \subseteq [A]^\omega$ is closed unbounded and $A \subseteq B$, then $\{x \in [B]^\omega : x \cap A \in C\}$ is closed unbounded in $[B]^\omega$.
- (d) If C is closed unbounded in $[B]^\omega$ and if $A \subseteq B$ then $\{x \cap A : x \in C\}$ contains a closed unbounded set in $[A]^\omega$.

If (P, \leq) is a notion of forcing then $B(P)$ will denote the corresponding complete Boolean algebra, and $V^P = V^{B(P)}$ the corresponding Boolean-valued model.

3. Cohen algebras

For every infinite cardinal κ , let \mathbf{C}_κ be the complete Boolean algebra that adjoins κ Cohen reals. In other words, \mathbf{C}_κ is the completion of Fr_κ , the free algebra on κ generators; more generally, let $\mathbf{C}_X = \overline{Fr_X}$ for any set X .

Definition 3.1. A Boolean algebra B is a *Cohen algebra* if $\bar{B} = \mathbf{C}_\kappa$ for some infinite cardinal κ .

In this section we prove the following characterization of Cohen algebras, that is a slight improvement of results of Koppelberg [10] and Bandlow [2].

Theorem 3.2. Let B be an infinite Boolean algebra of uniform density. B is a Cohen algebra if and only if the set $\{A \in [B]^\omega : A \leq_{\text{reg}} B\}$ contains a closed unbounded set C with the property

$$(*) \quad \text{if } A_1, A_2 \in C, \text{ then } \langle A_1 \cup A_2 \rangle \in C.$$

We remark that if B is a countable atomless algebra then B is a Cohen algebra, and the condition is satisfied trivially, since $C = \{B\}$ is (trivially) a closed unbounded subset of $[B]^\omega$.

We shall prove Theorem 3.2 in a sequence of lemmas. Throughout, we assume that B has uniform density.

Lemma 3.3. If B is a dense subalgebra of \mathbf{C}_κ then B has the property in Theorem 3.2.

Proof. Let C be the set of all countable subalgebras A of B with the property that there exists a countable set $S \subseteq \kappa$ such that A is a dense subset of $B \cap \mathbf{C}_S$ and $B \cap \mathbf{C}_S$ is a dense subset of \mathbf{C}_S . We shall prove that every $A \in C$ is a regular subalgebra of B , that C is closed unbounded and that $(*)$ is satisfied.

Let $A \in C$, with S being a witness. Since $B \cap \mathbf{C}_S$ is dense in \mathbf{C}_S and $\mathbf{C}_S \leq_{\text{reg}} \mathbf{C}_\kappa$, we have $B \cap \mathbf{C}_S \leq_{\text{reg}} \mathbf{C}_\kappa$, and because B is dense in \mathbf{C}_κ it follows that $B \cap \mathbf{C}_S \leq_{\text{reg}} B$. Since A is dense in $B \cap \mathbf{C}_S$, we have $A \leq_{\text{reg}} B$.

To show that C is unbounded, let $a \in B$ be arbitrary; we shall find an $A \in C$ such that $a \in A$. First, because \mathbf{C}_κ has the countable chain condition, there exists a countable

$S_0 \subseteq \kappa$ such that $a \in C_{S_0}$. Second, again using the countable chain condition, and because B is dense in C_κ , there exists a countable $S \subseteq \kappa$ such that $S_0 \subseteq S$ and that $B \cap C_S$ is dense in C_S . Finally, there is a countable subalgebra A of B such that $a \in A$ and that A is dense in $B \cap C_S$.

To show that C is closed, let $\{A_n\}_{n=0}^\infty$ be an increasing chain in C and let $A = \bigcup_{n=0}^\infty A_n$. Let $\{S_n\}_{n=0}^\infty$ be witnesses for the A_n . For each n , A_n is dense in C_{S_n} and a subalgebra of $C_{S_{n+1}}$; hence $S_n \subseteq S_{n+1}$. Since for each n , A_n is dense in C_{S_n} , it follows that A is dense in $\bigcup_{n=0}^\infty C_{S_n}$; but the latter is dense in C_S where $S = \bigcup_{n=0}^\infty S_n$. Hence, A is dense in $B \cap C_S$ which is dense in C_S .

Finally, we shall verify (*). Let $A_1, A_2 \in C$ and let S_1, S_2 be such that A_i is dense in C_{S_i} , for $i = 1, 2$. We shall show that $A = \langle A_1 \cup A_2 \rangle$ is dense in C_S where $S = S_1 \cup S_2$. Let $b \in C_S^+$ be arbitrary; we shall find $a_1 \in A_1$ and $a_2 \in A_2$ such that $0 \neq a_1 \cdot a_2 \leq b$.

The algebra C_S has as a dense set Cohen's forcing P_S , the set of all finite 0-1-functions on S . Let $p \in P_S$ be such that $p \leq b$. Let $p_1 = p \restriction S_1$ and $p_2 = p \restriction S_2$. First we find some $a_1 \in A_1^+$ such that $a_1 \leq p_1$, and then some $q_1 \in P_{S_1}$ such that $q_1 \leq a_1$. Let $q_2 = p_2 \restriction (S_2 \setminus S_1) \cup (q_1 \restriction S_2)$; we have $q_2 \in P_{S_2}$. Now we find some $a_2 \in A_2^+$ such that $a_2 \leq q_2$. We claim that $a_1 \cdot a_2 \neq 0$: there exists some $r_2 \in P_{S_2}$ such that $r_2 \leq a_2$, and then $r_2 \cup (q_1 \restriction (S_1 \setminus S_2)) \in P_S$ is below both a_1 and a_2 . \square

This lemma proves one direction of Theorem 3.2. For the other direction, we first prove that the property in Theorem 3.2 implies that B has the countable chain condition. In fact, we prove a stronger assertion and use it in Section 4.

Lemma 3.4. *Let B be a Boolean algebra such $\{A \in [B]^\omega : A \leq_{\text{reg}} B\}$ is stationary. Then B has the countable chain condition.*

Proof. Let W be a maximal antichain in B . Consider the model $M = \langle B, \leq, W \rangle$. There exists a stationary set of countable submodels $A \prec M$ such that $A \leq_{\text{reg}} B$. It follows that $W \cap A$ is a maximal antichain in A and therefore in B . Hence, $W = W \cap A$ and is countable. \square

The following lemma is due to Vladimirov [16, Lemma VII.3].

Lemma 3.5. *Let B be a complete Boolean algebra of uniform density and let A be a complete subalgebra of B of density less than the density of B . For every $v \in B$ there exists a $u \in B$ such that $a \cdot u \neq 0 \neq a - u$ for all $a \in A^+$, and $v \in A(u)$.*

We shall call u independent over A .

Lemma 3.6. *Let B have a closed unbounded set C of countable regular subalgebras, closed under $\langle A_1 \cup A_2 \rangle$. Let \mathcal{S} be a collection of all subalgebras of B of the form $\langle \bigcup S \rangle$ where $S \subseteq C$. Then every $A \in \mathcal{S}$ is a regular subalgebra of B .*

Proof. This is true for every finite $S \subset C$, and because C is closed, it is also true for every countable $S \subset C$. For an arbitrary S , let W be a maximal antichain in $\langle \bigcup S \rangle$. By Lemma 3.4, W is countable, and so $W \subset \langle \bigcup S_0 \rangle$ for some countable $S_0 \subseteq S$. Since $\langle \bigcup S_0 \rangle \leq_{\text{reg}} B$, W is a maximal antichain in B . Hence $\langle \bigcup S \rangle \leq_{\text{reg}} B$. \square

We shall now complete the proof of Theorem 3.2. Let B be an infinite Boolean algebra of uniform density $\kappa > \omega$ and let C be a closed unbounded subset of $[B]^\omega$ consisting of regular subalgebras of B and satisfying (*). Let

$$\mathcal{S} = \left\{ \left\langle \bigcup S \right\rangle : S \subseteq C \right\},$$

and let $D = \{d_\alpha : \alpha < \kappa\} \subset B^+$ be a dense subset of B .

If A_1 and A_2 are subalgebras of \bar{B} we say that A_1 and A_2 are *co-dense* if for every $a_1 \in A_1^+$ there exists some $a_2 \in A_2^+$ with $a_2 \leq a_1$, and for every $a_2 \in A_2^+$ there exists some $a_1 \in A_1^+$ with $a_1 \leq a_2$.

We shall construct, by induction on $\alpha < \kappa$, two continuous chains $G_0 \subset G_1 \subset \dots \subset G_\alpha \subset \dots$ and $B_0 \subset B_1 \subset \dots \subset B_\alpha \subset \dots$ such that

- (1) $B_\alpha \in \mathcal{S}$,
- (2) G_α is an independent subset of \bar{B} ,
- (3) $A_\alpha = \langle G_\alpha \rangle$ and B_α are co-dense,
- (4) $d_\alpha \in B_{\alpha+1}$,
- (5) $G_{\alpha+1} - G_\alpha$ is countable.

This will prove that B is a Cohen algebra, because by (4), $\bigcup_{\alpha < \kappa} B_\alpha$ is dense in B , hence $\bigcup_{\alpha < \kappa} A_\alpha$ is dense in \bar{B} , and it follows that $\bar{B} = \overline{\text{Fr}_G}$ where $G = \bigcup_{\alpha < \kappa} G_\alpha$.

At limit stages of the construction, we let $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$ and $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$; clearly, (1)–(3) are satisfied.

Thus, assume that we have constructed B_α and G_α , and find $G_{\alpha+1}$ and $B_{\alpha+1}$. Since A_α is dense in \bar{B}_α and \bar{B}_α is a complete subalgebra of \bar{B} , \bar{A}_α is a complete subalgebra of \bar{B} . Also, if $u_1, \dots, u_n \in \bar{B}$ then $\bar{A}_\alpha(u_1, \dots, u_n)$ is a complete algebra of \bar{B} .

Since $|A_\alpha| < \kappa$ and κ is the uniform density of B , we find, by Lemma 3.5, for every $b \in \bar{B}$ some $u \in \bar{B}$ independent over \bar{A}_α such that $b \in \bar{A}_\alpha(u)$. More generally, if $b, u_1, \dots, u_n \in \bar{B}$ then there exists some u independent over $\bar{A}_\alpha(u_1, \dots, u_n)$ such that $b \in \bar{A}_\alpha(u_1, \dots, u_n, u)$.

Given $u \in \bar{B}$, there exist countably many $\{b_n\}_{n=0}^\infty \subset B$ such that $\sum_{n=0}^\infty b_n = u$. Then there exists some $X \in C$ such that $\{b_n\}_n \subseteq X$ and so $\langle B_\alpha \cup X \rangle$ is dense in $\bar{A}_\alpha(u)$. Therefore, there exist a countable set $\{u_n\}_{n=0}^\infty \subset \bar{B}$, and some $B_{\alpha+1} \in \mathcal{S}$ such that $d_\alpha \in B_{\alpha+1}$, that $G_{\alpha+1} = G_\alpha \cup \{u_n\}_{n=0}^\infty$ is independent and that $A_{\alpha+1} = \langle G_{\alpha+1} \rangle$ and $B_{\alpha+1}$ are co-dense.

4. Semi-Cohen algebras

Motivated by the characterization of Cohen algebras, Fuchino and Jech introduced in 1992 the following property: a Boolean algebra B of uniform density is called semi-Cohen if it has a closed unbounded set of countable regular subalgebras.

We start with the following observation:

Theorem 4.1. *If B is semi-Cohen and if A is a regular subalgebra of B of uniform density then A is semi-Cohen.*

Proof. The family $[B]^\omega$ contains a closed unbounded subset of regular subalgebras of B . Since $A \leq_{\text{reg}} B$, there exists for every $b \in B^+$ some $a \in A^+$ such that $a - b$ does not have any $x \leq a - b$ in A^+ . Let $F: B^+ \rightarrow A^+$ be a function that to each $b \in B^+$ assigns such an $a \in A^+$. Let $C \subseteq [B]^\omega$ be a closed unbounded set of regular subalgebras, closed under F .

If $X \in C$ then $A \cap X \leq_{\text{reg}} X$ because X is closed under F . Every maximal antichain in $A \cap X$ is maximal in X , hence in B (because $X \leq_{\text{reg}} B$), hence in A . So $A \cap X \leq_{\text{reg}} A$.

There is a closed unbounded set $D \subseteq [A]^\omega$ such that $D \subseteq \{X \cap A: X \in C\}$. D witnesses that A is semi-Cohen. \square

Another consequence of Definition 1.1 is

Theorem 4.2. *If B is a semi-Cohen algebra and B has density \aleph_1 then B is a Cohen algebra.*

Proof. If B is semi-Cohen, and if $|B| = \aleph_1$ then B is Cohen, because every closed unbounded subset of $[B]^\omega$ contains a closed unbounded subset that is a chain under inclusion, hence closed under finite unions. If B has density \aleph_1 , let A be a dense subalgebra of B of size \aleph_1 . A is semi-Cohen by Theorem 4.1 and therefore Cohen, and since $\bar{A} = \bar{B}$, B is Cohen. \square

As a consequence, we have the following corollary: If B is a Cohen algebra and $A \leq_{\text{reg}} B$ has uniform density \aleph_1 , then A is a Cohen algebra. This fact was previously known to Koppelberg [9].

We shall now give equivalent characterizations of semi-Cohen algebras.

Theorem 4.3. *Let B be a Boolean algebra of uniform density. The following are equivalent:*

- (a) B is semi-Cohen.
- (b) $V^P \models B$ is Cohen, where P is the σ -closed collapse of $|B|$ onto \aleph_1 .
- (c) There exists a proper forcing P such that $V^P \models B$ is Cohen.
- (d) The second player has a winning strategy in the infinite game \mathcal{G} in which two players select in turn elements $a_0, b_0, a_1, b_1, \dots$ of the algebra B , and the second player wins if $\langle \{a_0, b_0, a_1, b_1, \dots\} \rangle \leq_{\text{reg}} B$.
- (e) I has a winning strategy in the following game \mathcal{H} : I plays elements a_i of B and II plays elements $b_i \leq a_i$. I wins iff $\sum b_i = 1$.

The game \mathcal{H} is introduced in [4] where the equivalence of (e) with semi-Cohen is proved.

First we state a corollary of this theorem:

Corollary 4.4. *An algebra B is semi-Cohen if and only if \bar{B} is semi-Cohen.*

Proof. One direction follows from Theorem 4.1, since $B \leq_{\text{reg}} \bar{B}$. Thus, assume that B is semi-Cohen, and show that \bar{B} is.

Let $A = \bar{B}$. Let P be the collapse (with countable conditions) of $|A|$ onto \aleph_1 . In V^P , B is dense in A and is a Cohen algebra; hence A is Cohen in V^P . Hence A is semi-Cohen in V . \square

To prove Theorem 4.3 we first prove that (a) and (b) are equivalent. Let P be the collapse of $|B|$ onto \aleph_1 with countable conditions. By Theorem 4.2, if B is semi-Cohen in V^P then B is Cohen. Thus, it suffices to show that $V^P \models$ “ B is semi-Cohen” if and only if B is semi-Cohen.

As P does not add new countable sets of ordinals, $[B]^\omega$ is the same in V^P as in V . Using property (c) of Lemma 2.2 we see that for every $A \in [B]^\omega$, $A \leq_{\text{reg}} B$ if and only if $V^P \models A \leq_{\text{reg}} B$.

Let S be the set of all countable regular subalgebras of B . If S contains a closed unbounded set C , then $V^P \models C$ is closed unbounded. Conversely, if S does not contain a closed unbounded set, then because P is proper, $V^P \models S$ does not contain a closed unbounded set. Hence, B is semi-Cohen if and only if $V^P \models B$ is semi-Cohen.

A similar argument establishes the equivalence of (a) with (c): As (c) follows from (b), let us assume that P is proper and $V^P \models B$ is semi-Cohen. Let S be the set of all countable regular subalgebras. If B were not semi-Cohen then $[B]^\omega - S$ would be stationary, therefore stationary in V^P , contrary to the assumption that B is semi-Cohen in V^P .

To see that (a) and (d) are equivalent, consider the game \mathcal{G} . If $F: B^{<\omega} \rightarrow B$ is a winning strategy for the second player then the set C of all countable subalgebras of B closed under F is closed unbounded and all of its elements are regular subalgebras. Conversely, if $[B]^\omega$ has a closed unbounded set of regular subalgebras then it is easy to find a winning strategy for the second player, using Lemma 2.4(a).

For the convenience of the reader we outline the proof of the equivalence of (a) and (e), which is the content of Theorem 1.6 of [4]. If I has a winning strategy σ in \mathcal{H} , then the club C of all countable subalgebras of B closed under σ consists of regular subalgebras: if $A \in C$ were not regular then there would be some maximal antichain W in A and some $b \in B^+$ incompatible with W . Then II can play moves $b_i \leq a_i$ within A incompatible with b . In the end, $\Sigma b_i \perp b$, contradicting the assumption on σ .

Conversely, if there is a club C consisting of regular subalgebras of B , I wins the game by catching her tail: make sure that $A = \{a_i\}_i = \{b_i\}_i \in C$. Then in A , $\Sigma b_i = 1$, and by regularity, $\Sigma b_i = 1$ in B . \square

Next we shall prove that the class of semi-Cohen algebras is closed under unions of regular chains. In the next Section we show that this is not necessarily true for Cohen algebras.

Theorem 4.5. *Let $\{B_\alpha : \alpha < \vartheta\}$ be a continuous increasing chain of semi-Cohen Boolean algebras such that $B_\alpha \leq_{\text{reg}} B_{\alpha+1}$ for every α . Then $\bigcup_{\alpha < \vartheta} B_\alpha$ is semi-Cohen, provided it has uniform density.*

Let us remark that uniform density need not be preserved by limits of chains. The theorem is analogous to Ščepin's Theorem on openly generated Boolean algebras [5].

Proof. Let $\{B_\alpha : \alpha < \vartheta\}$ be a regular continuous chain of semi-Cohen algebras with limit $B = B_\vartheta$. By Lemma 2.2 we can choose *pseudo-projections* $\pi_{\alpha\beta} : B_\beta \rightarrow B_\alpha$ for all $\alpha \leq \beta \leq \vartheta$, i.e. functions such that for every $b \in B_\beta^+$ and for every $x \in B_\alpha^+$, if $x \leq \pi_{\alpha\beta}(b)$ then $x \cdot b \neq 0$.

Let λ be a sufficiently large regular cardinal. We will show that for every countable elementary submodel M of $\langle H_\lambda, \in, B, \{B_\alpha\}_\alpha, \{\pi_{\alpha\beta}\}_{\alpha\beta} \rangle$, $B \cap M$ is a regular subalgebra of B . This will show that B is semi-Cohen.

Fix the model M and let $\gamma = \sup M \cap \vartheta$. For every $b \in B^+$ we must find an $a \in B \cap M$ such that every nonzero $x \in B \cap M$ below a is compatible with b . Fix some $b \in B^+$. Let $b_\gamma = \pi_{\gamma\vartheta}(b)$; there is some $\alpha \in \gamma \cap M$ such that $b_\gamma \in B_\alpha$. Since B_α is semi-Cohen, $B_\alpha \cap M$ is a regular subalgebra of B_α and there is an $a \in B_\alpha \cap M$ such that every nonzero $z \in B_\alpha \cap M$ below a is compatible with b_γ . We claim that this a works.

To show that, let $x \in B \cap M$ be below a ; we will prove that $x \cdot b \neq 0$. Let $y = \pi_{x\vartheta}(x)$; $a \cdot y$ is a nonzero element of $B_\alpha \cap M$, $a \cdot y \leq a$ and so $a \cdot y \cdot b_\gamma \neq 0$ by the choice of a . Since $y = \pi_{x\vartheta}(x)$ and $a \cdot y \cdot b_\gamma \leq y$, we conclude that $a \cdot y \cdot b_\gamma \cdot x \neq 0$. By the same reasoning $a \cdot y \cdot b_\gamma \cdot x \cdot b \neq 0$ and therefore $x \cdot b \neq 0$, as needed. \square

The next theorem lists other closure properties of the class of semi-Cohen algebras.

Theorem 4.6. (a) *If A and B are semi-Cohen algebras then $A \times B$ is semi-Cohen.*

(b) *If A is semi-Cohen and $V^A \models \dot{B}$ is semi-Cohen then $A * \dot{B}$ is semi-Cohen.*

(c) *If B is semi-Cohen and W is an extension of the universe then $W \models B$ is semi-Cohen.*

By (b) and by Theorem 4.5, any finite support iteration of semi-Cohen algebras is semi-Cohen, provided it has uniform density. Note also that the class of all Cohen algebras also has properties (a)–(c).

Proof. (a) is an easy consequence of property (b) of Theorem 4.3. A similar argument proves (b): Let P be the σ -closed collapse of $|A * \dot{B}|$, and note that \dot{B} is Cohen in $(V^P)^A$; thus $A * \dot{B}$ is Cohen in V^P .

To prove (c), we use property (e) of Theorem 4.3. Let σ be a winning strategy for the first player in the game \mathcal{H} . The following statement is a theorem of ZFC:

σ is a winning strategy if and only if for every $b \in B$ the tree $T_b = \{g : g \text{ is a finite play of the game } \mathcal{H} \text{ according to } \sigma \text{ such that II's answers are incompatible with } b\}$ is well-founded.

For example, if σ is not a winning strategy then there is a play $a_0, b_0, a_1, b_1, \dots$ of the game \mathcal{H} according to σ such that $\sum b_i \neq 1$. Consequently, if b is incompatible with $\sum b_i$ then this play is an infinite path through the tree T_b .

Now the wellfoundedness of trees T_b in (*) is absolute between transitive models of set theory. Therefore any winning strategy σ for I remains a winning strategy in any extension W of the universe and so $W \models B$ is semi-Cohen. \square

The forcing properties of semi-Cohen algebras are much like those of Cohen algebras; e.g. only Cohen reals are added. We include the following lemma that will be needed in Section 5.

Lemma 4.7. *Semi-Cohen forcings do not add new branches to trees of height ω_1 .*

Proof. For contradiction, let P be semi-Cohen, let T be a tree of height ω_1 and let $p \in P$ force that \dot{b} is a new branch through T . Choose a sufficiently large regular cardinal λ and a countable elementary submodel M of H_λ that contains P, p, \dot{b}, T . Let $\gamma = M \cap \omega_1$ and choose a condition $q \leq p$ and some t at level γ of T such that $q \Vdash t \in \dot{b}$. Since P is semi-Cohen, the poset $P \cap M$ is regular in P and there is an $r \in P \cap M$ such that any extension of r in M is compatible with q . But then, $c = \{s \in T : \text{there is some extension of } r \text{ forcing } s \text{ into } \dot{b}\}$ is a branch through T and so $r \Vdash \dot{b} = c$, contradicting our assumption. \square

It should be remarked that the lemma can be easily adapted for the two generalizations of semi-Cohenness in Sections 6 and 7.

The last result of this section is the following theorem:

Theorem 4.8. *Every semi-Cohen complete Boolean algebra B of density κ contains C_κ as a complete subalgebra.*

Proof. Let B be a semi-Cohen complete Boolean algebra of uncountable density κ . (The case when $\kappa = \aleph_0$ is trivial.) There exists a function $F : B^{<\omega} \rightarrow B$ such that every countable subalgebra of B closed under F is a regular subalgebra. First we claim that every subalgebra of B closed under F is a regular subalgebra. Let A be a subalgebra of B closed under F and let W be a maximal antichain in A . Let A_1 be the smallest subalgebra of B closed under F such that $W \subset A_1$. Then A_1 is a subalgebra of A , and W is a maximal antichain in A_1 . As A_1 is countable, it is a regular subalgebra of B and so W is a maximal antichain in B .

Using F , we can find a continuous chain of regular subalgebras of B of size less than κ , whose union is dense in B . Thus there exist complete Boolean subalgebras B_α of B , $\alpha < \kappa$, each of density $< \kappa$, such that $B_\alpha \subset B_\beta$ whenever $\alpha < \beta$, and for every limit λ , $\bigcup_{\alpha < \lambda} B_\alpha$ is dense in B_λ . As B has uniform density κ , we may assume that for every α , $\bigvee^{B_\alpha} \dot{B}_{\alpha+1} \models B_{\alpha+1} : B_\alpha$ is nontrivial. Thus, $B = B(P)$ where P is the finite support iteration of $\langle \dot{Q}_\alpha : \alpha < \kappa \rangle$, with $\dot{Q}_\alpha = B_{\alpha+1} : B_\alpha$.

We now claim that P (in fact every finite support iteration of nontrivial forcings) embeds \mathbf{C}_κ . For each α , let $\dot{a}_\alpha \in V^{B_\alpha}$ be a name for an element of $B_{\alpha+1} : B_\alpha$ such that $V^{B_\alpha} \models 0 \neq \dot{a}_\alpha \neq 1$.

If $p \in P$ then we can find some stronger q with the property that for every $\alpha \in \text{support}(q)$, q decides $\dot{a}_\alpha \in \dot{G}$, where \dot{G} is the name for the generic ultrafilter. Let Q be the dense subset of P consisting of all q with such a property. For every $q \in Q$, let $h(q)$ be the 0–1 function on $\text{support}(q)$ such that $h(q)(\alpha) = 1$ just in case $q \Vdash \dot{a}_\alpha \in \dot{G}$.

Let P_κ be the (version of) Cohen forcing consisting of all finite 0–1 functions f with $\text{dom}(f) \subset \kappa$; we have $B(P_\kappa) = \mathbf{C}_\kappa$. The function h maps Q onto P_κ and has the property that for every $q \in Q$ and every $f \supset h(q)$ there is some $q' \leq q$ such that $h(q') \supseteq f$. Hence, h witnesses that P_κ embeds regularly into Q , and so \mathbf{C}_κ embeds into B as a complete subalgebra. \square

5. Examples

By Theorem 4.2, there is only one complete semi-Cohen algebra of uniform density \aleph_1 , namely the Cohen algebra \mathbf{C}_{ω_1} . In this section we show that for higher densities there are other, significantly different, semi-Cohen algebras.

Since semi-Cohenness is inherited by complete subalgebras, we can find semi-Cohen algebras looking at complete subalgebras of \mathbf{C}_{ω_2} . But are not all of these again isomorphic to \mathbf{C}_{ω_2} ? [8, 9]. The following result from [11] shows that this is not the case and thereby provides a new kind of semi-Cohen algebras.

Theorem 5.1 (Koppelberg and Shelah [11]). *There is a complete subalgebra of \mathbf{C}_{ω_2} of uniform density \aleph_2 which is not isomorphic to \mathbf{C}_{ω_2} .*

Proof. Let us define a partially ordered set P as follows:

$$P = \{z : z \text{ is a function, } \text{dom}(z) \in [\omega_2]^{<\omega}, \text{ran}(z) \subseteq \omega^{<\omega}\}.$$

We order P by $z \leq w$ if z is a coordinatewise extension of w and for $\alpha \neq \beta$ both in $\text{dom}(w)$ if $n \in \text{dom}(z(\alpha) - w(\alpha))$ and $n \in \text{dom}(z(\beta))$ then $z(\alpha)(n) \neq z(\beta)(n)$. Thus, P is the forcing for adding a sequence of ω_2 eventually different reals. If $G \subseteq P$ is a generic filter then G can be decoded from a P -generic sequence of functions $\langle f_\alpha : \alpha < \omega_2 \rangle \subseteq \omega^\omega$, where $f_\alpha = \bigcup \{z(\alpha) : z \in G\}$.

Let $B = B(P)$. We shall prove that the algebra B witnesses the statement of the theorem. First, a helpful observation.

Lemma 5.2. *The algebra B is isomorphic to the completion of the poset R , a finite support iteration*

$$R = \langle R_\alpha : \alpha \leq \omega_2, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$$

such that for every $\alpha < \omega_2$ the poset R_α forces:

- (1) The Q_α generic is given by a real \dot{f}_α ,
- (2) $Q_\alpha = \{\langle s, a \rangle : s \in \omega^{<\omega}, a \in [\alpha]^{<\omega}\}$ ordered by $\langle t, b \rangle \leq \langle s, a \rangle$ if $s \subseteq t$, $a \subseteq b$ and $\forall n \in \text{dom}(t - s) \forall \beta \in a \ t(n) \neq f_\beta(n)$,
- (3) the Q_α -term \dot{f}_α is defined by $Q_\alpha \Vdash \dot{f}_\alpha = \bigcup \{s \in \omega^{<\omega} : \langle s, \emptyset \rangle \in \dot{G}\}$, where \dot{G} is the name for the Q_α -generic filter,
- (4) Q_α is a separative partial order of uniform density $|\alpha| + \aleph_0$.

Proof. Left to the reader. \square

Following the lemma, we shall represent the algebra B as $B = \bigcup_{\alpha < \omega_2} B_\alpha$, where $B_\alpha = B(R_\alpha)$. The following two lemmas finish the proof of Theorem 5.1.

Lemma 5.3. B is not isomorphic to C_{ω_2} .

Lemma 5.4. B can be completely embedded into C_{ω_2} .

Proof of Lemma 5.3. For contradiction, assume that $h: B \rightarrow C_{\omega_2}$ is an isomorphism. Then by a simple closure argument, there is an α , $\omega_1 < \alpha < \omega_2$, such that $h''B_\alpha = C_\alpha$. We reach a contradiction working in V^{C_α} . Let Q_α, \dot{f}_α be as defined in Lemma 5.2; we have $Q_\alpha \in V^{C_\alpha}$. Since the residue forcing $C_{\omega_2}:C_\alpha$ is Cohen, every real added by it comes from a C_ω -extension of V^{C_α} . In particular, the real \dot{f}_α comes from such an extension. Since \dot{f}_α determines a Q_α -generic filter (Lemma 5.2(1)) we have $B(Q_\alpha) \leq_{\text{reg}} C_\omega$. This is a contradiction, since $B(Q_\alpha)$ has uniform density \aleph_1 (Lemma 5.2(4)) and C_ω has uniform density \aleph_0 . \square

Proof of Lemma 5.4. We define another forcing S by $S = \{w : w \text{ is a finite function with } \text{dom}(w) \subseteq (\omega_2 \times 2) \cup \{\omega_2\} \text{ such that}$

- (1) $\langle \alpha, 0 \rangle \in \text{dom}(w) \Rightarrow w(\alpha, 0) \in \omega^{<\omega}$,
- (2) $\langle \alpha, 1 \rangle \in \text{dom}(w) \Rightarrow w(\alpha, 1) \in \omega$,
- (3) $\omega_2 \in \text{dom}(w) \Rightarrow w(\omega_2)$ is a finite function from $\omega^{<\omega}$ to ω which is one-to-one on each ω^n , $n \in \omega\}$.

We order S by $w \leq v$ if $\text{dom}(v) \subseteq \text{dom}(w)$ and

- (1) $\langle \alpha, 0 \rangle \in \text{dom}(v) \Rightarrow v(\alpha, 0) \subseteq w(\alpha, 0)$,
- (2) $\langle \alpha, 1 \rangle \in \text{dom}(v) \Rightarrow v(\alpha, 1) = w(\alpha, 1)$,
- (3) $\omega_2 \in \text{dom}(v) \Rightarrow v(\omega_2) \subseteq w(\omega_2)$.

Let $G \subseteq S$ be a generic filter. The forcing is designed to add by finite conditions functions k, h, g_α ($\alpha < \omega_2$):

- (1) $g_\alpha = \bigcup \{w(\langle \alpha, 0 \rangle) : w \in G\} \in \omega^\omega$,
- (2) $h: \omega_2 \rightarrow \omega$ defined by $h(\alpha) = n$ if $\langle \langle \alpha, 1 \rangle, n \rangle \in G$,
- (3) $k = \bigcup \{w(\omega_2) : w \in G\}$ is a function from $\omega^{<\omega}$ which is one-to-one on each ω^n .

Note that $g_\alpha: \alpha < \omega_2$ are mutually generic Cohen reals.

Obviously, $B(S) = C_{\omega_2}$ and it is enough to find a complete embedding of $B(P) = B$ into $B(S)$. We show how to read off a P -generic sequence $\langle \dot{f}_\alpha : \alpha < \omega_2 \rangle$ from a generic

filter $G \subseteq S$ and define the associated projection $pr: B(S) \rightarrow B$ on a dense set $D \subseteq S$. The verification is left to the reader.

So let $G \subseteq S$ be generic, $k, h, g_\alpha \in V[G]$ as above. In $V[G]$, we define $f_\alpha: \omega \rightarrow \omega$ by $f_\alpha \upharpoonright h(\alpha) = g_\alpha \upharpoonright h(\alpha)$, $f_\alpha(n) = k(g_\alpha \upharpoonright (n+1))$ for $n \geq h(\alpha)$.

To define the projection, we let D be the set of all $w \in S$ such that (1) $\langle \alpha, 0 \rangle \in \text{dom}(w)$ iff $\langle \alpha, 1 \rangle \in \text{dom}(w)$, (2) the sequences $w(\alpha, 0)$, $\langle \alpha, 0 \rangle \in \text{dom}(w)$, are pairwise distinct and there is an integer $n \in \omega$ such that $w(\alpha, 0) \in \omega^n$, and $w(\alpha, 1) < n$, and (3) $\omega_2 \in \text{dom}(w)$ and $\text{dom}(w(\omega_2)) = \{s \in \omega^{\leq n} : s \text{ is an initial segment of some } w(\alpha, 0)\}$.

It is immediate that $D \subseteq S$ is dense. We define $pr(w)$, for $w \in D$, to be the $z \in P$ for which:

- (1) $\text{dom}(z) = \{\alpha : \langle \alpha, 0 \rangle \in \text{dom}(w)\}$,
- (2) $z(\alpha) \upharpoonright w(\alpha, 1) = w(\alpha, 0) \upharpoonright w(\alpha, 1)$ and $z(\alpha)(m) = w(\omega_2)(w(\alpha, 0)) \upharpoonright (m+1)$ for $w(\alpha, 1) \leq m < \text{dom}(w(\alpha, 0))$. \square

Semi-Cohen algebras with more complicated properties can be produced using similar methods. For example, the forcing from [6] is similar to the one we just described. The following argument describes a semi-Cohen algebra B of uniform density \mathfrak{c}^+ that cannot be embedded into a Cohen algebra.

Let P be a forcing for adding a modulo finite increasing chain of functions from ω into the rationals \mathcal{Q} of length \mathfrak{c}^+ . Then $B = B(P)$ is semi-Cohen and since no Cohen algebra adds such a chain [12], B does not embed into any \mathbf{C}_κ . To be more precise, set $P = \{z : \text{dom}(z) \in [\mathfrak{c}^+]^{<\omega} \text{ and there is an integer } n_z \in \omega \text{ such that for } \alpha \in \text{dom}(z) \text{ } z(\alpha) \in \mathcal{Q}^{n_z}\}$. The ordering is defined by $z \leq w$ if z is a coordinatewise extension of w and for $\alpha < \beta$ both in $\text{dom}(w)$ and $n_w \leq m < n_z$ we have $z(\alpha)(m) < z(\beta)(m)$. Thus, if $G \subseteq P$ is a generic filter and $\alpha < \mathfrak{c}^+$, we can set $f_\alpha: \omega \rightarrow \mathcal{Q}$ to be $\bigcup \{f : \langle \alpha, f \rangle \in G\}$ and we will have $\alpha < \beta < \mathfrak{c}^+ \Rightarrow f_\alpha(n) < f_\beta(n)$ for all but finitely many $n \in \omega$. The only thing left to prove is the semi-Cohenness of $B = B(P)$. For $X \subseteq \mathfrak{c}^+$ define $P_X = \{z \in P : \text{dom}(z) \subseteq X\}$. Then P_X is a regular subposet of P , with the projection $pr: P \rightarrow P_X$ defined by $pr(z) = z \upharpoonright X$. Thus, $\{P_X : X \in [\mathfrak{c}^+]^\omega\}$ is a club set of regular subposets of P and consequently, $B = B(P)$ is semi-Cohen.

The last example in this section gives a sharper result and provides a completely different semi-Cohen algebra.

Theorem 5.5. *There is a semi-Cohen algebra B of uniform density \aleph_2 which cannot be completely embedded into a Cohen algebra.*

Proof. Let P be Tennenbaum's forcing for adding a Souslin tree with finite conditions [14]. So $P = \{\langle t, <_t \rangle : t \text{ is a finite subset of } \omega_1 \text{ and } <_t \text{ is a tree order on } t \text{ respecting the ordering of ordinals}\}$; we order P by reverse extension. Let the P -term for a Souslin tree \dot{T} be defined by $P \Vdash \dot{T} = \langle \omega_1, <_T \rangle$, where $<_T = \bigcup \{<_t : \langle t, <_t \rangle \in G\}$.

Lemma 5.6. (1) $B(P) \cong \mathbf{C}_{\omega_1}$,

(2) the completion of $P * (\dot{T} \text{ upside down})$ is isomorphic to \mathbf{C}_{ω_1} ,

(3) $B(P * \dot{S}) \cong \mathbf{C}_{\omega_1}$, where \dot{S} is a P -name for the forcing that makes \dot{T} special, cf. [3].

Proof. (1) is a consequence of both (2) and (3). The proofs of (2) and (3) are similar; we prove (3).

All we have to show by Theorem 4.2 is that the algebra $B(P * \dot{S})$ is semi-Cohen. The forcing $P * \dot{S}$ has a dense subset $D = \{\langle t, <_t, f \rangle : \langle t, <_t \rangle \in P \text{ and } f : t \rightarrow \omega \text{ is a function such that } \alpha <_t \beta \Rightarrow f(\alpha) \neq f(\beta)\}$, ordered by reverse extension. Let $D_\alpha = \{\langle t, <_t, f \rangle \in D : t \leq \alpha\}$.

It is enough to show that for a limit ordinal $\alpha \in \omega_1$ the poset D_α is a regular subposet of D . Then $\{D_\alpha : \alpha \in \omega_1 \text{ limit}\}$ is a club subset of regular subposets of D , proving the semi-Cohenness of $B(P * \dot{S})$. So fix $\alpha \in \omega_1$ limit. We define the projection $pr : E \rightarrow D_\alpha$ for a dense set $E \subseteq D$. Let $E = \{\langle t, <_t, f \rangle \in D : \forall \beta \in t \cap \alpha \text{ if there is a } \gamma \in t \text{ such that } \beta <_t \gamma \text{ and } f(\gamma) = n \text{ then there is } \gamma \in t \cap \alpha \text{ such that } \beta <_t \gamma \text{ and } f(\gamma) = n\}$.

Obviously, $E \subseteq D$ is dense and a function $pr : E \rightarrow D_\alpha$ defined by $pr\langle t, <_t, f \rangle = \langle t \cap \alpha, <_t \upharpoonright \alpha^2, f \upharpoonright \alpha \rangle$ is a projection. \square

By a similar argument as in the lemma, the algebra $B(P * \dot{X} * \dot{Q})$ is isomorphic to \mathbf{C}_{ω_1} , where \dot{X} is (forced to be) a four-element algebra with two atoms $x, 1 - x$ and $1 \Vdash_P x \Vdash_{\dot{X}} \text{“}\dot{Q} \text{ is the forcing } \dot{T} \text{ upside down”}$, $1 \Vdash_P 1 - x \Vdash_{\dot{X}} \text{“}\dot{Q} \text{ is the } \dot{T}\text{-specialization forcing”}$.

Finally, we are in a position to define the complete algebra B witnessing the statement of the theorem. We let $B = B(R)$ where R is the finite support iteration

$$R = \langle R_\alpha : \alpha \leq \omega_2, \dot{U}_\alpha : \alpha < \omega_2 \rangle,$$

where $U_0 = \mathbf{C}_{\omega_1}$, for $0 < \alpha < \omega_2$ we have $B(R_\alpha) \cong \mathbf{C}_{\omega_1}$ and at stage α , $0 < \alpha < \omega_2$, we find an isomorphism $i_\alpha : B(R_\alpha) \rightarrow B(P)$, get an R_α -name \dot{T}_α for a Souslin tree and set $\dot{U}_\alpha = X_\alpha * \dot{Q}_\alpha$ as above, that is, X_α has two atoms x_α and $1 - x_\alpha$ and $x_\alpha \Vdash_{X_\alpha} \text{“}\dot{Q}_\alpha \text{ is the forcing with } \dot{T}_\alpha \text{ upside down”}$ and $1 - x_\alpha \Vdash_{X_\alpha} \text{“}\dot{Q}_\alpha \text{ is the } \dot{T}_\alpha\text{-specialization forcing”}$.

We have $B = \bigcup_{\alpha < \omega_2} B_\alpha$, where $B_\alpha = B(R_\alpha)$.

Lemma 5.7. B is a semi-Cohen algebra of uniform density \aleph_2 .

Proof. By induction on $\alpha \in \omega_2$ we prove that B_α is a semi-Cohen algebra of uniform density \aleph_1 , i.e. $B_\alpha \cong \mathbf{C}_{\omega_1}$. At limits steps, we use Theorem 4.2, and at successor steps, we apply the observation following the previous lemma. Then lemma follows immediately from Theorem 4.2. \square

Lemma 5.8. B cannot be completely embedded into a Cohen algebra.

Proof. Assume, for contradiction, that it can be. Then B can be embedded into \mathbf{C}_{ω_2} and we can consider B as a complete subalgebra of \mathbf{C}_{ω_2} . The following can be proved by a simple closure argument.

Claim 5.9. *The set $C = \{\alpha \in \omega_2 : B_\alpha \subseteq C_\alpha\} \subseteq \omega_2$ is closed unbounded.*

Even better, we have

Claim 5.10. *The set $D = \{\alpha \in C : \text{for no condition } p \in C_\alpha, p \text{ decides the statement } \dot{x}_\alpha \in G\}$ contains a club in ω_2 .*

Proof. First, define a B -name \dot{f} for a function from ω_2 to $\{0, 1\}$ by $B \Vdash \dot{f}(\alpha) = 1$ iff $\dot{x}_\alpha \in \dot{G}$. Obviously,

(*) $B \Vdash \dot{f} \text{ is a } C_{\omega_2}\text{-generic function over the ground model}.$

Now assume that the set $C - D$ is stationary. For any $\alpha \in C - D$, let $p_\alpha \in C_\alpha$ decide the statement “ $\dot{x}_\alpha \in G$ ”. There exist a stationary set $S \subseteq C - D$ and a $p \in C_{\omega_2}$ such that for every $\alpha \in S$ we have $p = p_\alpha$. Then $p \Vdash \dot{f} \restriction S$ belongs to the ground model”, contradicting (*) since the set S is infinite. Hence, D contains a closed unbounded set. \square

Now fix an ordinal $\alpha \in D$. We have $B_\alpha \leq_{\text{reg}} C_\alpha$ and so \dot{T}_α is a C_α -name for an ω_1 -tree. We reach a contradiction:

Case I: There is $p \in C_\alpha$ such that $p \Vdash_{C_\alpha} \dot{T}_\alpha$ has a cofinal branch”. Since $\alpha \in D$, we have $p \cdot (1 - x_\alpha) \neq 0$ in C_{ω_2} . Also, $1 - x_\alpha \Vdash_B \dot{T}_\alpha$ is a special tree”. By upwards absoluteness, $p \cdot (1 - x_\alpha) \Vdash_{C_{\omega_2}} \dot{T}_\alpha$ is a special tree with a cofinal branch”. It follows that ω_1 must be collapsed, contradicting c.c.c. of C_{ω_2} .

Case II: $C_\alpha \Vdash \dot{T}_\alpha$ has no cofinal branches”. Then $C_{\omega_2} \Vdash \dot{T}_\alpha$ has no cofinal branches” since the residue forcing $C_{\omega_2} : C_\alpha$ is Cohen and as such does not add branches to Aronszajn trees (Lemma 4.7). However, $x_\alpha \Vdash_B \dot{T}_\alpha$ has a cofinal branch”, so by upwards absoluteness $x_\alpha \Vdash_{C_{\omega_2}} \dot{T}_\alpha$ has a cofinal branch”, contradiction.

This completes the proof of Theorem 5.5. \square

The following example shows that Theorem 4.5 fails for Cohen algebras:

Theorem 5.11. *There exists an increasing ω -chain of the Cohen algebras B_n , with $B_n \leq_{\text{reg}} B_{n+1}$, whose union is not a Cohen algebra.*

Proof. Let P be the forcing notion from Theorem 5.1. A generic G on P yields ω_2 functions $\{g_\alpha : \alpha < \omega_2\}$ from ω into ω . For each n and each $\alpha < \omega_2$, let $g_{\alpha,n}(k) = g_\alpha(k) \bmod 2^n$, and let $G_n = \{g_{\alpha,n} : \alpha < \omega_2\}$.

We have $V[G] = V[\{G_n : n \in \omega\}]$, and for all $n < m$, $V[G_n] \subset V[G_m]$.

Let P_n be the forcing with finite conditions that adjoins ω_2 functions from ω into 2^n . $B(P_n)$ is a Cohen algebra; we claim that G_n is a generic on P_n :

If D is an open dense set in P_n , let E be the set of all conditions whose projection belongs to D . As E is dense in P , it follows that G_n is generic on P .

Hence, $B(P)$ is the limit of a regular ω -chain of Cohen algebras. \square

The limit $B(P)$ in the above proof is embeddable into a Cohen algebra. By taking instead the forcing P described in the comments following the proof of Theorem 5.1, we obtain a regular ω -chain of Cohen algebras whose limit B is not embeddable into a Cohen algebra. In this case, each B_n has density $\geq (2^{\aleph_0})^+$.

6. Pseudo-Cohen algebras

Looking for other classes of algebras which share some of the properties of \mathbf{C}_κ , we arrive at the following generalization of Definition 1.1.

Definition 6.1. An algebra B of uniform density is *pseudo-Cohen* if the set $S = \{A \in [B]^\omega : A \leq_{\text{reg}} B\}$ is stationary.

By Lemma 3.4, pseudo-Cohen algebras are c.c.c. and all reals added by them are in a Cohen-generic extension of the ground model. However, the class of pseudo-Cohen algebras does not have most of the closure properties of the semi-Cohen class. While it is closed under regular subalgebras (Theorem 6.2 below), it is not closed under products or iterations, since the nature of the witness set S may vary.

Theorem 6.2. *If B is pseudo-Cohen and if A is a regular subalgebra of uniform density then A is pseudo-Cohen.*

Proof. Follows closely the proof of Theorem 4.1. \square

The following generalizes Theorem 4.3:

Theorem 6.3. *The following are equivalent:*

- (a) B is a pseudo-Cohen algebra.
- (b) there is an \aleph_0 -distributive forcing P such that $P \Vdash "B \text{ is Cohen}"$.
- (c) the first player does not have a winning strategy in the game \mathcal{G} .

Proof. (a) implies (b): Let B be a pseudo-Cohen algebra as witnessed by a set S and let P be the standard forcing for shooting a club through S , namely $P = \{f : f \text{ is a function from some } \alpha + 1 < \omega_1 \text{ to } S \text{ which is increasing and continuous with respect to } \subset\}$, ordered by extension. The forcing P collapses $|B|$ to \aleph_1 and is \aleph_0 -distributive. In V^P the algebra B has size \aleph_1 and a club subset of regular subalgebras, therefore, it is Cohen by Theorem 4.2.

(b) implies (a): Let $P \Vdash "B \text{ is a Cohen algebra}"$ for some \aleph_0 -distributive forcing P . Fix a function $f : B^{<\omega} \rightarrow B$. We must produce a countable regular subalgebra A of B closed under f . Such an algebra certainly exists in the generic extension by P , where B is Cohen. But since P does not add any countable subsets of B , such a subalgebra A exists already in the ground model.

(a) implies (c): Let σ be a strategy for the player I in the game \mathcal{G} associated with a pseudo-Cohen algebra B . Since the witness set S is stationary, we can fix a large

regular cardinal θ and a countable submodel $M \prec \langle H_\theta, \in, B, \sigma \rangle$ with $M \cap B \leq_{\text{reg}} B$. Since the model M is closed under σ , there is a play of \mathcal{G} in which player I uses σ , and in which the second player picks all the elements of $M \cap B$. This shows that σ is not a winning strategy for the player I.

(c) implies (a): Left to the reader. \square

Pseudo-Cohenness is preserved by proper forcing extensions:

Proposition 6.4. *If B is pseudo-Cohen and W is a proper-forcing extension of the universe then $W \models B$ is pseudo-Cohen.*

Proof. The witness S from Definition 6.1 remains stationary in W . \square

We complete this section by giving an example of a pseudo-Cohen algebra that is not semi-Cohen. Fix a set $\{l_\alpha : \alpha < \omega_1 \text{ limit}\}$, where $l_\alpha : \omega \rightarrow \alpha$ is an increasing sequence of ordinals converging to α . Fix $S \subset \omega_1$ co-stationary and let $P_S = \{\langle f, s \rangle : f \text{ is a finite function from } \omega_1 \text{ to } \{0, 1\} \text{ and } s \text{ is a finite set of countable ordinals}\}$ ordered by $\langle g, t \rangle \leq \langle f, s \rangle$ if $f \subset g$, $s \subset t$ and for every $\beta \in \text{dom}(g - f)$ and every $\alpha \in s \cap S$, if $\beta \in \text{ran}(l_\alpha)$ then $g(\beta) = 0$.

In the generic extension by P_S , let $F : \omega_1 \rightarrow \{0, 1\}$ be defined by $F = \bigcup \{f : \langle f, \emptyset \rangle \text{ is in the generic filter}\}$. By the definition of the ordering on P_S , a countable limit ordinal α is in S just in the case when only finitely many β 's in $\text{ran}(l_\alpha)$ have $F(\beta) = 1$.

We claim that $B(P_S)$ is a pseudo-Cohen algebra. To show this, it is enough to prove that for every $\alpha \in \omega_1 - S$ the poset $R_\alpha = \{\langle f, s \rangle \in P_S : s \subset \alpha, \text{dom}(f) \subset \alpha\} \subset P_S$ is a regular subposet of P_S . Fix an arbitrary condition $\langle f, s \rangle \in P_S$. We shall produce a condition $\langle g, t \rangle \in R_\alpha$ such that any extension of it in R_α is compatible with $\langle f, s \rangle$ in P_S . Let $x = s \cap S - \alpha$ and $y = \{\beta < \alpha : \beta \notin \text{dom}(f) \text{ and } \exists \xi \in x \beta \in \text{ran}(l_\xi)\}$. Thus, the set y is finite, because $\alpha \notin S$. We let g be the extension of $f \upharpoonright \alpha$ to $\alpha \cup y$ such that $g = 0$ on y , and $s = t \cap \alpha$. It is easy to see that $\langle g, s \rangle$ is as required.

Thus, $B(P_S)$ is a pseudo-Cohen algebra. If the set $S \subset \omega_1$ is chosen stationary co-stationary, it is not difficult to prove that $B(P_S)$ is not semi-Cohen and $B(P_S) \times B(P_{\omega_1 - S})$ is not pseudo-Cohen. Thus we have:

Theorem 6.5. *There exists a pseudo-Cohen algebra that is not semi-Cohen.*

We can also show that the algebra constructed by Veličković in [15] for adding a Kurepa tree by a c.c.c. forcing from \square_{ω_2} is also pseudo-Cohen.

7. Potentially Cohen algebras

Generalizing properties (c) in Theorem 4.3 and (b) in Theorem 6.2 we arrive at the following notion:

Definition 7.1. An algebra B of uniform density is *potentially Cohen* if there is an ω_1 -preserving forcing notion such that $\mathcal{Q} \Vdash "B \text{ is a Cohen algebra}"$.

Every pseudo-Cohen algebra is potentially Cohen. In this section we investigate the converse. We prove:

Theorem 7.2. (CH) *Every potentially Cohen algebra is pseudo-Cohen.*

Theorem 7.3. (MA + \neg CH) *The measure algebra is potentially Cohen.*

Notice that the measure algebra is not pseudo-Cohen, since it does not add Cohen reals. Thus, Theorem 7.3 proves the necessity of the CH assumption in Theorem 7.2 and shows that in general, potential Cohenness of an algebra is a considerably weaker property. Indeed, the only significant properties of an algebra B implied by its potential Cohenness without further assumptions seem to be included in the following simple lemma.

Lemma 7.4. *If B is a potentially Cohen algebra, then B is c.c.c. and not \aleph_0 -distributive.*

Proof. Let B be potentially Cohen and $Q \Vdash$ “ B is Cohen”, for some ω_1 -preserving forcing Q . If $A \subseteq B$ is an uncountable antichain, then A remains uncountable in V^Q , contradicting c.c.c. of B . So the algebra B is c.c.c. If B were \aleph_0 -distributive, then there would be a Souslin tree T such that $B(T) \leq_{\text{reg}} \bar{B}$. Then $Q \Vdash$ “ $B(T) \leq_{\text{reg}} \bar{B} \cong \mathbf{C}_\kappa$ for some κ ” and since the uniform density of $B(T)$ is \aleph_1 , by Theorem 4.2 $Q \Vdash B(T) \cong \mathbf{C}_{\omega_1}$. So in the generic extension by Q , forcing with the ω_1 -tree T adds reals. Consequently $Q \Vdash$ “ $B(T)$ collapses ω_1 and so $B(T) \not\cong \mathbf{C}_{\omega_1}$ ”, contradiction. \square

Proof of Theorem 7.2. Assume that B is a potentially Cohen poset as witnessed by an ω_1 -preserving forcing Q . First, we treat the special case when $|B| = \aleph_1$. Let us enumerate $B = \{p_\alpha : \alpha < \omega_1\}$. We shall show that the set $S = \{\beta < \omega_1 : B_\beta = \{p_\alpha : \alpha < \beta\} \leq_{\text{reg}} B\}$ is a stationary subset of ω_1 , which proves the pseudo-Cohenness of the poset B . Indeed, in the forcing extension V^Q we will have $S = \{\beta < \omega_1 : B_\beta = \{p_\alpha : \alpha < \beta\} \leq_{\text{reg}} B\}$ and since $V^Q \models$ “ B is a Cohen algebra”, the set S contains a closed unbounded subset in V^Q . Consequently, S is stationary in V .

In the case of a poset B of higher cardinality, we have to prove that the set $S = \{A \in [B]^\omega : A \leq_{\text{reg}} B\}$ is stationary. So let $f : B^{<\omega} \rightarrow B$ be an arbitrary function. We shall produce an element of the set S closed under f .

Lemma 7.5. (CH) *There is a regular subalgebra A_1 of B of size \aleph_1 closed under the function f .*

Proof. Choose a large regular cardinal κ and a submodel $M \prec H_\kappa$ of size \aleph_1 such that $f, B \in M$ and $[M]^\omega \subset M$. This is possible since the Continuum Hypothesis holds. Now any maximal antichain X of the poset $B \cap M$ is an antichain in B and so is countable. Therefore, $X \in M$ and by elementarity X is a maximal antichain of B . Consequently, the algebra $A_1 = M \cap B \leq_{\text{reg}} B$ is as required. \square

Let A_1 be as in Lemma 7.5. Now $Q \Vdash "A_1 \leq_{\text{reg}} B \leq_{\text{reg}} \bar{B}"$ and so A_1 is a regular subalgebra of a Cohen algebra and by Theorem 4.2, \bar{A}_1 is a Cohen algebra itself". Thus, A_1 is potentially Cohen. By the first part of the proof, A_1 is pseudo-Cohen and there is a countable subposet $A \leq_{\text{reg}} A_1$ closed under the function f . So, $A \leq_{\text{reg}} A_1 \leq_{\text{reg}} B$ is as required. \square

Proof of Theorem 7.3. Assume $MA + \mathfrak{c} > \aleph_1$. The theorem will follow from these two lemmas.

Lemma 7.6. *(MA + $\neg CH$) There is an ω_1 -preserving forcing Q such that $Q \Vdash "cf(c^V) = \omega"$.*

Lemma 7.7. *(MA + $\neg CH$) For any family $\{B_\alpha : \alpha < \kappa\}$ of positive Borel sets, where $\kappa < \mathfrak{c}$, there are positive closed sets $C_i : i < \omega$ such that $\forall \alpha < \kappa \exists i < \omega C_i \subseteq B_\alpha$.*

Let a forcing Q be as in Lemma 7.6. Then $Q \Vdash$ “the measure algebra B from the ground model has a countable dense set”. To see this, in the ground model we enumerate B as $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$, choosing Borel representatives for each equivalence class. By Lemma 7.7, for each ordinal $\kappa < \mathfrak{c}$ there are closed positive sets $C_i^k : i < \omega$ such that $\forall \alpha < \kappa \exists i < \omega C_i^k \subseteq B_\alpha$. Now if $Q \Vdash "\langle \kappa_j : j < \omega \rangle$ is a sequence converging to c^V ", then $Q \Vdash "\{C_i^{\kappa_j} : i, j < \omega\}$ is a countable dense subset of B ". Thus, the forcing Q witnesses the fact that the measure algebra B is potentially Cohen.

Proof of Lemma 7.6. Let $\mathfrak{I} = \{A \subset \mathcal{P}_{\aleph_2}(\mathfrak{c}) : \exists y \in \mathcal{P}_{\aleph_2}(\mathfrak{c}) \forall x \in A \ y \not\subset x\}$ be the ideal of bounded subsets of $\mathcal{P}_{\aleph_2}(\mathfrak{c})$. Since $\mathfrak{c} > \aleph_1$, the ideal \mathfrak{I} is proper and \aleph_2 -complete. We define the forcing Q to be the set of all trees T of finite sequences of elements of $\mathcal{P}_{\aleph_2}(\mathfrak{c})$ such that the tree T has a trunk t and for each sequence $s \in T$ extending t the set $A_s = \{y \in \mathcal{P}_{\aleph_2}(\mathfrak{c}) : s \smallfrown \langle y \rangle \in T\}$ is not in the ideal \mathfrak{I} . In the spirit of Namba forcing proofs [13], one can argue that Q preserves \aleph_1 . Also, if $G \subset Q$ is a generic filter, then in $V[G]$, the set $\bigcup G$ is an ω -sequence of sets which are of cardinality \aleph_1 in the ground model and whose union exhausts all of c^V . Since by $MA + \neg CH$, $cf(\mathfrak{c}) > \omega_1$ in V , the suprema of these sets are smaller than c^V and converge to c^V . \square

Proof of Lemma 7.7. Let us recall one of the definitions of the amoeba forcing:

$A = \{\langle \mathcal{O}, \delta \rangle : 0 < \delta \leq 1 \text{ is a real number and}$

$\mathcal{O} \subset [0, 1] \text{ is an open set of measure } < \delta\}$.

The order is defined by $\langle \mathcal{O}, \delta \rangle \geq \langle \mathcal{P}, \gamma \rangle$ if $\gamma \leq \delta$ and $\mathcal{O} \subset \mathcal{P}$. The forcing A is known to be σ -linked and so a finite support product A^ω of ω copies of A is c.c.c.

Fix a family $\{B_\alpha : \alpha < \kappa\}$ of positive Borel sets of reals, where $\kappa < \mathfrak{c}$. For $\alpha < \kappa$ we define sets $D_\alpha \subset A^\omega$ by $p \in D_\alpha$ iff $\exists i < \omega p(i) = \langle \mathcal{O}, \delta \rangle$ for some real $\delta \in (0, 1]$ and an open set \mathcal{O} such that $[0, 1] - \mathcal{O} \subset B_\alpha$. It is easy to see that for $\alpha < \kappa$ the set $D_\alpha \subset A^\omega$ is open dense: if $p \in A^\omega$ and $\alpha < \kappa$, then one can choose an integer i with $i \notin \text{support}(p)$

and a closed positive set $C \subseteq B_x$. Then $q = p \cup \{\langle i, \langle [0, 1] - C, 1 \rangle \rangle\}$ is a condition in D_x which is smaller than p .

Also, for an integer $i \in \omega$ define a dense subset $E_i \subset A^\omega$ by $p \in E_i$ if $p(i) = \langle \emptyset, \delta \rangle$ for some $\delta < 1$.

By Martin's Axiom there is a filter $G \subset A^\omega$ meeting all the dense sets $D_x : \alpha < \kappa$ and $E_i : i < \omega$. For $i < \omega$ we define a closed set $C_i = [0, 1] - \bigcup \{\emptyset : \langle i, \langle \emptyset, 1 \rangle \rangle \in G\}$. Since the filter G meets all the E_i 's, the sets C_i are positive. Since the filter meets all the sets D_α , for every $\alpha < \kappa$ there is an integer i such that $C_i \subset B_\alpha$. \square

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